Criteria for Sign Regularity of Sets of Matrices*

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ABSTRACT

We consider matrix intervals with respect to a certain "checkerboard" partial ordering. We show that every real matrix contained in a matrix interval is sign-regular if two special matrices taken from that matrix interval are sign-regular.

1. NOTATION AND DEFINITIONS

Let $A=(\alpha_{ij})$ be an $n\times n$ matrix. The submatrix of A formed from rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$ will be denoted by

$$A\begin{bmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k \end{bmatrix} \quad \text{or} \quad A[i_1,\dots,i_k; j_1,\dots,j_k],$$

and its determinant by

$$A\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k \end{pmatrix} \quad \text{or} \quad A(i_1,\dots,i_k; j_1,\dots,j_k).$$

The matrix A is called sign-regular[1] [4, p. 47] if all nonzero kth-order minors of A have the same sign $\sigma_k(A)$, $k=1(1)n$ (we set $\sigma_k(A):=0$ if all kth-order minors vanish). If in this case $\sigma_k(A)\geqslant 0$, $k=1(1)n$, we call A totally nonnegative. We say that A is strictly sign-regular if A is sign-regular and all minors of A are nonzero. The definitions and basic properties of these

*Dedicated to Professor R. Krawczyk on the occasion of his sixtieth birthday.

[1] It should be noted that the German equivalent of "sign-regular," namely "zeichenregulär," is used in a slightly different sense by Gantmacher and Krein in [1, p. 86].

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matrices were given in [1]; see also [4]. The matrix A will be said to be inverse-positive if $A^{-1}=(\alpha'_{ij})$ exists and $\alpha'_{ij}\geqslant 0$, $i,j=1(1)n$.

We define $A^*\in\mathbb{R}^{n\times n}$ by $A^*:=DAD$, where $D:=\operatorname{diag}(1,-1,\dots,(-1)^{n+1})$. The transformation $*$ is usually called the "checkerboard transformation." As usual, $A\leqslant B$ and $A<B$ for $A,B\in\mathbb{R}^{n\times n}$ will be understood entrywise. Let $A\leqslant^* B$ and $A<^* B$, respectively, if $A^*\leqslant B^*$ and $A^*<B^*$, respectively. The set of matrix intervals with respect to the partial ordering $\leqslant^*$ will be denoted by $\mathbb{I}(\mathbb{R}^{n\times n})$, and its elements by $[A]=[\underline{A},\overline{A}]$, $\underline{A}=(\underline{\alpha}_{ij})$, $\overline{A}=(\overline{\alpha}_{ij})$; $[B]=[\underline{B},\overline{B}]$, etc.

A matrix interval $[A]$ will said to be nonsingular [respectively, (strictly) sign-regular, totally nonnegative] if all the real matrices contained in $[A]$ are nonsingular [respectively, (strictly) sign-regular, totally nonnegative].

2. PRELIMINARY RESULTS

LEMMA 1. *If $[A]\in\mathbb{I}(\mathbb{R}^{n\times n})$ with $\sigma_n:=\sigma_n(\underline{A})=\sigma_n(\overline{A})\neq 0$ and all the nonzero $(n-1)$-minors of $\underline{A}$ and $\overline{A}$ have the same sign σ_{n-1}, then $\sigma_n\sigma_{n-1}A^*$ is inverse-positive for any $A\in[A]$; in particular $[A]$ is nonsingular.*

Proof. The proof can be easily given by using the formula for the inverse of a matrix in terms of its adjoint matrix and by exploiting the following well-known statement (e.g. [5, Corollary 3.5]): A matrix B is inverse-positive iff there exist inverse-positive matrices $\underline{B}$, $\overline{B}$ such that $\underline{B}\leqslant B\leqslant\overline{B}$. ∎

REMARK. Lemma 1 can be used to bound the set of all solutions of special systems of linear equations with intervals as coefficients [3].

The following lemma shows that the approximation of sign-regular matrices by strictly sign-regular matrices which is given in Satz 17 in [1, p. 311] can be done from above and from below with respect to the partial ordering $\leqslant$.

LEMMA 2. *Let the $n\times n$ matrix $A=(\alpha_{ij})$ be of rank r and sign-regular. For every $\varepsilon>0$ there exists a strictly sign-regular $n\times n$ matrix $C=(\gamma_{ij})$ such that for arbitrary $\tau_k\in\{-1,1\}$ we have*

$$\sigma_k(C)=\begin{cases}\sigma_k(A), & k=1(1)r,\\ \tau_k, & k=r+1(1)n,\end{cases}\tag{1}$$

and

$$0<\sigma_1(C)(\gamma_{ij}-\alpha_{ij})<\varepsilon, \qquad i,j=1(1)n. \tag{2}$$

Proof. We follow the proof given in [1, pp. 308, 310–311]. If $n=1$, the statement holds trivially. We suppose now $n\geqslant 2$. Let e_i be the ith unit vector of $\mathbb{R}^n$, i.e. $e_i:=\delta_i$, where δ_i is the usual Kronecker delta. In case $r=0$ replace A by the matrix $\sigma_1(C)\varepsilon' e_1 e_1^t$, where $\varepsilon':=\varepsilon/3$. If A has exactly one nonzero coefficient, then replace A by $A+\sigma_1(A)\varepsilon' xy^t$, where $x,y\in\mathbb{R}^n$ are defined as follows:

$$\begin{array}{llll} \text{in case } \sigma_2(C)=1, & x:=y:=e_1 & & \text{if } \alpha_{11}=0,\\ & x:=y:=e_n & & \text{else;}\\ \text{in case } \sigma_2(C)=-1, & x:=e_1, & y:=e_n & \text{if } \alpha_{1n}=0,\\ & x:=e_n, & y:=e_1 & \text{else.} \end{array}$$

Put $F(\eta):=(\exp[-\eta(i-j)^2])$, $i,j=1(1)n$, where $\eta>0$. It is known [1, p. 89] that every minor of $F(\eta)$ is positive. Obviously $F(\eta)\to I$ (identity matrix) as $\eta\to\infty$. From the Cauchy-Binet formula [1, pp. 11–12] it follows that the matrix $B=(\beta_{ij})$ defined by $B:=F(\eta)AF(\eta)$ has only nonzero kth-order minors and satisfies

$$\sigma_k(B)=\sigma_k(A), \qquad k=1(1)r, \tag{3}$$

and obviously for sufficiently small $\eta>0$

$$0<\sigma_1(A)(\beta_{ij}-\alpha_{ij})<\varepsilon'. \tag{4}$$

If $r=n$, the matrix B fulfills (1) and (2). If $r<n$ we set $\delta:=\min\{|\beta_{ij}-\alpha_{ij}|\mid i,j=1(1)n\}$. The approximation process of the proof of Satz 17 in [1, p. 311] yields a matrix $C=(\gamma_{ij})$ satisfying

$$\sigma_k(C)=\begin{cases}\sigma_k(B), & k=1(1)r,\\ \tau_k, & k=r+1(1)n,\end{cases} \tag{5}$$

and

$$|\gamma_{ij}-\beta_{ij}|<\delta<\varepsilon', \qquad i,j=1(1)n. \tag{6}$$

By (3) and (5), the matrix C fulfills the equation (1), and from (4) and (6) we obtain (2). ∎

3. CRITERIA FOR SIGN REGULARITY

THEOREM 1. *Let* $[A]\in\mathbb{I}(\mathbb{R}^{n\times n})$. *Then*

(i) $[A]$ *is strictly sign-regular*

if and only if

(ii) $\underline{A}$ *and* $\overline{A}$ *are strictly sign-regular and* $\sigma_k(\underline{A})=\sigma_k(\overline{A})$, $k=1(1)n$.

Proof. The implication (i) $\Rightarrow$ (ii) holds by the continuity of the determinant. We suppose now (ii) and $A\in[A]$. The case $n=1$ is trivial, and we consider the case $n>1$. Because of Theorem 3.3 in [4, p. 60] it suffices to show that all the minors of A composed from consecutive rows and columns have the same strict sign σ_k.

Let $p\in\{0,1,\dots,n-1\}$, and assume without loss of generality $p\geqslant 1$. Then we have for $i,j=1(1)n-p$

$$\underline{A}\begin{bmatrix} i,\dots,i+p \\ j,\dots,j+p\end{bmatrix}\begin{Bmatrix}\leqslant *\\ *\geqslant\end{Bmatrix}A\begin{bmatrix} i,\dots,i+p \\ j,\dots,j+p\end{bmatrix}\begin{Bmatrix}\leqslant *\\ *\geqslant\end{Bmatrix}\overline{A}\begin{bmatrix} i,\dots,i+p \\ j,\dots,j+p\end{bmatrix}$$
$$\text{if}\quad i+j \text{ is } \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}.$$

The determinants of these two submatrices of $\underline{A}$ and $\overline{A}$ have the same strict sign σ_{p+1}. The same holds for the pth-order minors. It follows from Lemma 1 that

$$A\begin{bmatrix} i,\dots,i+p \\ j,\dots,j+p\end{bmatrix}\quad \text{is nonsingular.}$$

Finally, for $q,m\in\{1,2,\dots,n-p\}$

$$\operatorname{sign} A\begin{pmatrix} i,\dots,i+p \\ j,\dots,j+p\end{pmatrix}=\sigma_{p+1}=\operatorname{sign} A\begin{pmatrix} q,\dots,q+p \\ m,\dots,m+p\end{pmatrix}. \qquad \blacksquare$$

THEOREM 2. *Let* $[A]=[\underline{A},\overline{A}]\in\mathbb{I}(\mathbb{R}^{n\times n})$ *satisfy*

either $\forall i,j\in\{1,\dots,n\}\quad \underline{\alpha}_{ij}=\overline{\alpha}_{ij}\ \Rightarrow\ i+j$ *is even,*

or $\forall i,j\in\{1,\dots,n\}\quad \underline{\alpha}_{ij}=\overline{\alpha}_{ij}\ \Rightarrow\ i+j$ *is odd.*

Then

(i) $[A]$ *is sign-regular (respectively, sign-regular and nonsingular) and* $\sigma_k(A)\sigma_k(B)\geqslant 0$, $k=1(1)n$, *for all* $A, B\in[A]$

if and only if

(ii) $\underline{A}$ *and* $\overline{A}$ *are sign-regular (respectively, sign-regular and nonsingular) and* $\sigma_k(\underline{A})\sigma_k(\overline{A})\geqslant 0$, $k=1(1)n$.

Proof.[2] The statement involving both sign-regular and nonsingular follows immediately from the other part by Lemma 1.

It suffices to show (ii)⇒(i) in case $n>1$. Without loss of generality we suppose that $\sigma_1(\underline{A}), \sigma_1(\overline{A})\geqslant 0$. By Lemma 2 there exist two sequences of strictly sign-regular matrices, say $\{\underline{C}^{(\nu)}\},\{\overline{C}^{(\nu)}\}$, $\underline{C}^{(\nu)}=(\underline{\gamma}_{ij}^{(\nu)})$, $\overline{C}^{(\nu)}=(\overline{\gamma}_{ij}^{(\nu)})$, $\nu=1,2,\ldots$, satisfying

$$\left.\begin{array}{ll}\lim\underline{C}^{(\nu)}=\underline{A}, & \underline{\alpha}_{ij}<\underline{\gamma}_{ij}^{(\nu)}\\ \lim\overline{C}^{(\nu)}=\overline{A}, & \overline{\alpha}_{ij}<\overline{\gamma}_{ij}^{(\nu)}\end{array}\right\}\qquad i,j=1(1)n,\quad \nu=1,2,\ldots,$$

and

$$\sigma_k(\underline{C}^{(\nu)})=\sigma_k(\overline{C}^{(\mu)}),\qquad \nu,\mu=1,2,\ldots,\quad k=1(1)n.$$

We will show that these sequences contain subsequences, say $\{\underline{A}^{(\nu)}\},\{\overline{A}^{(\nu)}\}$, with

$$\left.\begin{array}{l}\underline{A}^{(\nu)}\leqslant^{*}\overline{A}^{(\nu)},\\ \sigma_k(\underline{A}^{(\nu)})=\sigma_k(\overline{A}^{(\nu)}),\quad k=1(1)n\end{array}\right\}\qquad \nu=1,2,\ldots. \tag{7}$$

For brevity, let

$$\Gamma:=\left\{(i,j)\in\{1,\ldots,n\}^2\,|\,\underline{\alpha}_{ij}=\overline{\alpha}_{ij}\right\}.$$

For $(i,j)\notin\Gamma$ and $\nu_0\leqslant\nu$, ν_0 sufficiently large, we have

$$\underline{\alpha}_{ij}<\underline{\gamma}_{ij}^{(\nu)}<\overline{\alpha}_{ij}<\overline{\gamma}_{ij}^{(\nu)}\qquad\text{if}\quad i+j\text{ is even},$$

$$\overline{\alpha}_{ij}<\overline{\gamma}_{ij}^{(\nu)}<\underline{\alpha}_{ij}<\underline{\gamma}_{ij}^{(\nu)}\qquad\text{if}\quad i+j\text{ is odd}.$$

[2]The statement of Satz 4.5 (i.e. Theorem 2 formulated for the special case of totally nonnegative matrices) in [2] is incomplete. The assumption (∗) is missing there.

Without loss of generality we suppose that for every $(i,j)\in\Gamma$, $i+j$ is even. Then there exists $\nu_1 \geqslant \nu_0$ such that

$$\underline{\alpha}_{ij}=\bar{\alpha}_{ij}<\underline{\gamma}_{ij}^{(\nu_1)}\leqslant\bar{\gamma}_{ij}^{(\nu_0)}\qquad\text{for}\quad(i,j)\in\Gamma.$$

Let $\overline{A}^{(1)}:=\overline{C}^{(\nu_0)}$, $\underline{A}^{(1)}:=\underline{C}^{(\nu_1)}$, $\overline{A}^{(2)}:=\overline{C}^{(\nu_1)}$. We continue in this manner and obtain the subsequences $\{\underline{A}^{(\nu)}\},\{\overline{A}^{(\nu)}\}$ satisfying (7).

From Theorem 1 it follows that the interval $[\underline{A}^{(\nu)},\overline{A}^{(\nu)}]$ is strictly sign-regular. By taking the limit, the proof of the theorem is completed. ■

REMARK. If $[A]$ is sign-regular and nonsingular, we have already $\sigma_k(A)\sigma_k(B)\geqslant 0$, $k=1(1)n$, for all $A,B\in[A]$. This follows from the continuity of the function

$$\sum_{\substack{i_1<\cdots<i_k\\ j_1<\cdots<j_k}} A\begin{pmatrix} i_1,\ldots,i_k\\ j_1,\ldots,j_k\end{pmatrix}\qquad\text{for fixed}\quad k\in\{1,\ldots,n\}.$$

In a similar way one shows

THEOREM 3. *Let* $[A]=[\underline{A},\overline{A}]\in\mathbb{I}(\mathbb{R}^{n\times n})$ *be sign-regular and* $\sigma_k(A)\sigma_k(B)\geqslant 0$, $k=1(1)n$, *for all* $A,B\in[A]$. *Furthermore, suppose* $\underline{A}<^*\overline{A}$. *Then any* $A\in[A]$ *with* $\underline{A}<^*A<^*\overline{A}$ *is strictly sign-regular.*

The matrix interval $[A]=[\underline{A},\overline{A}]:=[e_4e_1^t,I]\in\mathbb{I}(\mathbb{R}^{4\times 4})$, where e_4 and e_1 are respectively the fourth and the first unit vector of $\mathbb{R}^4$ and where I is the identity matrix, has totally nonnegative boundary matrices $\underline{A}$ and $\overline{A}$. But $[A]$ contains the matrix $A:=I+e_4e_1^t$, which is nonsingular, but not totally nonnegative because $A(2,4;1,2)<0$. Thus this example shows that the statement of Theorem 2 involving merely sign regularity is not true without the assumption $(*)$. Note that $\overline{A}$ is nonsingular.

CONJECTURE. The matrix interval $[A]$ is nonsingular and totally nonnegative iff $\underline{A}$ and $\overline{A}$ are nonsingular and totally nonnegative.

We have tried hard, without success, to prove this conjecture. However, the conjecture is true for $n\leqslant 5$, so that the construction of a counterexample is nontrivial. Furthermore, the conjecture is true in the following special case.[3]

[3]The conjecture is also true if $\underline{A},\overline{A}\in T(r,s)$ with $T(r,s)$ from [6], i.e., $\underline{A},\overline{A}$ are band matrices with special nonsingular totally nonnegative submatrices.

THEOREM 4. *Let $[A]=[\underline{A},\bar{A}]\in\mathbb{I}(\mathbb{R}^{n\times n})$ be tridiagonal, i.e., $0=\underline{\alpha}_{ij}=\bar{\alpha}_{ij}$ if $|i-j|>1$, $i,j=1(1)n$. Then $[A]$ is nonsingular and totally nonnegative iff $\underline{A}$ and $\bar{A}$ are nonsingular and totally nonnegative.*

Proof. Let the matrix interval $[A]$ be tridiagonal, let $\underline{A}$ and $\bar{A}$ be nonsingular and totally nonnegative, and let $A\in[A]$. Then $A\geqslant 0$. By Hadamard's inequality (see Satz 8 in [1, p. 108]) every leading principal minor of $\underline{A}$ and $\bar{A}$ [i.e. $\underline{A}(1,\dots,i;1,\dots,i)$ and $\bar{A}(1,\dots,i;1,\dots,i)$] is positive. Therefore, any leading principal submatrix of $\underline{A}$ and $\bar{A}$ is totally nonnegative and nonsingular. Then by Lemma 1 and the continuity of the determinant, it follows that also the leading principal minors of A are positive. By a criterion in [1, p. 94] A is totally nonnegative. ■

We conclude this section with a theorem concerning the existence of totally nonnegative matrices.

THEOREM 5. *Suppose that $n\geqslant 3$, A is a nonsingular totally nonnegative matrix, and there exist $i\in\{1,\dots,n-2\}$, $k\in\{2,3,\dots,n-i\}$ such that*

$$A\begin{pmatrix} i+1,\dots,i+k \\ i,\dots,i+k-1 \end{pmatrix}=0 \tag{8}$$

or

$$A\begin{pmatrix} i,\dots,i+k-1 \\ i+1,\dots,i+k \end{pmatrix}=0. \tag{9}$$

Then there exists no totally nonnegative matrix B with $A<^*B$.

Proof. Suppose $A=(\alpha_{ij})$ is a nonsingular totally nonnegative matrix and (8) holds. By Corollary 9.1 in [4, p. 89], every principal minor of A is positive. Suppose $B=(\beta_{ij})$ is a totally nonnegative matrix with $A<^*B$. Then $\gamma:=|\beta_{i+k,i}-\alpha_{i+k,i}|>0$. From Lemma II in [7] it follows that

$$\det\left(A\begin{bmatrix} i+1,\dots,i+k \\ i,\dots,i+k-1 \end{bmatrix}+(-1)^{2i+k}\gamma e_k e_1^t\right)$$

$$=-\gamma A\begin{pmatrix} i+1,\dots,i+k-1 \\ i+1,\dots,i+k-1 \end{pmatrix}<0,$$

where e_1 and e_k are respectively the first and the kth unit vector of $\mathbb{R}^k$. The matrix on the left-hand side of the equation is a submatrix of a matrix contained in $[A, B]$. By Theorem 2, we have thus arrived at a contradiction. If (9) holds we proceed in a similar way. ■

EXAMPLE. Let the real numbers ψ_i and χ_i, $i=1(1)n$, have the same strict sign and

$$\frac{\psi_1}{\chi_1} < \frac{\psi_2}{\chi_2} < \cdots < \frac{\psi_n}{\chi_n}.$$

Then the *Green's matrix* [4, p. 110] $G=(\gamma_{ij})$ defined by

$$\gamma_{ij} := \begin{cases} \psi_i \chi_j & \text{if} \quad i \leqslant j, \\ \psi_j \chi_i & \text{if} \quad i \geqslant j \end{cases}$$

is nonsingular and totally nonnegative. Moreover, by [1, p. 90] Equations (8) and (9) hold.

This paper is an extension of some results given in the author's dissertation [2] and in [3].

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Received 6 May 1981; revised 21 October 1981